Financial Market Equilibrium with Bounded Awareness*

Ani Guerdjikova† and John Quiggin‡
University of Grenoble   University of Queensland

December 2, 2020

Abstract

We consider an infinite-horizon economy with differential awareness. For such economies, we propose an equilibrium concept which requires agents' consumption to be measurable with respect to the individual awareness partitions. We illustrate how the obtained equilibrium allocations observationally differ from those in economies with full awareness. In particular, economies with differential awareness can exhibit (i) lack of insurance against idiosyncratic risk; (ii) partial insurance against aggregate risk; (iii) biased state prices even when beliefs are correct and (iv) overpricing of assets which pay on events with low aggregate payoffs. We next adapt the results of Guerdjikova and Quiggin (2019 a) to show that agents with different levels of awareness can survive and influence prices in the limit. Moreover, differential awareness can lead to belief heterogeneity even in the limit. This is in contrast with the classical result of Blume and Easley (2006) stating that only agents with beliefs closest to the truth can survive. Finally, we examine the welfare implications of bounded awareness. If an increase in awareness comes at the cost of wrong beliefs over the larger state-space, bounded awareness can be welfare-improving, both from an individual and from a social point of view.

1 Introduction

The standard model of financial markets is one in which the set of assets spans the space of state-contingent consumption possibilities for all agents. Equilib-

*This research was supported by IUF, ANR-17-CE26-0003, Labex MME-DII.

†GAEL, University Grenoble Alpes, INRAE, CNRS, Grenoble INP, IUF, 1241 rue des Résidences, 38400 Saint Martin d'Hères, France, Tel.: +33 4 56 52 85 78, Email: ani.guerdjikova@univ-grenoble-alpes.fr.

‡University of Queensland, School of Economics, Brisbane 4072 Australia, j.quiggin@uq.edu.au

rium is generated by rational agents choosing over state-contingent consumption paths to maximize utility.

In reality, however, the set of financial instruments is too complex for any individual to comprehend. Most investors hold relatively simple portfolios, commonly consisting of a mix of bonds and mutual funds. The set of contingencies that might affect consumption allocations is even more complex. Hence, it is necessary to reconsider the notion of financial market equilibrium

Guerdjikova and Quiggin (2019 a) address this problem by considering agents who are exogenously constrained to invest in only a limited portfolio of assets. An alternative is to derive these constraints from the fact that real agents, unlike those in the standard model, are boundedly rational. However, while unboundedly rational agents are all alike (up to differences in utility functions and discount rates), there are many different concepts of bounded rationality, and not all are useful in modelling portfolio choices.

Developments in the theory of unawareness deal with a form of bounded rationality that is relevant to the problem of financial market equilibrium with constrained asset choices. Given the complexity of the world , boundedly rational agents can only consider a more limited set of possibilities than those embodied in the full state space. The resulting more limited state space, about which agents can reason, may be derived in two ways.

On the one hand, as in Heifetz, Meier and Schipper (2006) and Epstein, Marinacci and Seo (2007), agents may fail to distinguish between distinct states of nature, and may treat them like a single state. We refer to this as coarse awareness.

Alternatively, agents may fail to consider some possibilities at all. Grant and Quiggin (2013 a) use the term 'restricted awareness' to describe the associated state-space representation of bounded awareness, in which the state space considered by the agent is a proper subset $\Omega \subseteq \Sigma$ of the full state space.

The contrast between coarse unawareness and restricted unawareness may be illustrated with reference to a simple portfolio choice. Suppose there are two assets, a bond and an equity, and two states of the world, boom and slump. The bond yields 1 in both states, while the equity yields $1 + q$ $(q > 0)$ in the boom state and 0 in the slump state. Assume that $\pi_B (1 + q) > 1$ where π_B is the probability of the boom state. Thus, a fully aware investor who is neither risk neutral nor maximally risk-averse and has smooth preferences will choose to hold at least some of both assets.

First consider coarse awareness. An agent with coarse awareness of financial markets might fail to distinguish between the two states of the world. Such an agent cannot be aware of equity investments, assuming she perceives the state-contingent returns accurately. Hence, she will invest only in bonds[1].

Next consider restricted awareness. An agent who fails to consider one of the two states will place probability 1 on the other. If the investor considers only

[1] Guiso and Jappelli (2005) collect data about the awareness of Italian investors of different investment opportunities. Not surprisingly, they find that households fail to invest in assets of which they are unaware.

the (boom) slump state, she will regard the equity (bond) asset as dominant, and invest only in that asset.

As this example illustrates, restricted awareness will, in general, lead agents to hold incorrect beliefs about the probability of relevant events. This suggests the possibility of an analysis of survival in financial markets based entirely on bounded awareness. In this paper, however, in order to focus on the case of coarse awareness, we will take differential probability beliefs as primitive.

We will assume that agents cannot trade on partitions of the state space finer than those consistent with their awareness. Equivalently, they can only hold assets yielding payoffs measurable with respect to their awareness. An important implication is that an analysis of markets with awareness requires us to explicitly consider such constraints on agents' choices due to their limited perception of the world. When awareness varies across agents, this leads to a significant departure from common models of market incompleteness, in that agents behave as if they face different financial constraints.

Our primary focus will be thus on the way in which coarse awareness acts as a constraint on the portfolios and trades available to agents. We will show the existence of an equilibrium in an economy where agents have coarse awareness. Conditions under which asset markets are rich enough to yield this equilibrium are given in Guerdjikova and Quiggin (2019 b). We further show how this equilibrium may be distinguished observationally from one with full awareness.

We next consider the implications for survival in financial markets. A large literature, beginning with Blume and Easley (1992, 2006) is devoted to the idea that markets favor the best-informed and most rational traders. Trades in a financial market may be seen as 'betting one's beliefs' about the relative probabilities of different states of nature, and the resulting returns on assets. Over time, traders who correctly judge these probabilities and make rational investment choices based on their beliefs will accumulate wealth at the expense of others. In the limit, only these rational well-informed traders will survive, and market prices will reflect their beliefs. Boundedly rational traders with incorrect beliefs will not survive. These results may be expressed in terms of the 'survival index', the sum of the discount factor and the Kullback-Leibler distance between the agent's probabilistic beliefs about states and the true probability distribution.

This argument is intuitively appealing, and the central result can be derived under relatively weak conditions. However, the argument requires a high degree of sophistication on the part of agents who are, by hypothesis, boundedly rational, in that they have incorrect probability beliefs. It seems preferable to make bounded rationality explicit.

The analysis in this paper is similar, in some respects, to that of Guerdjikova and Quiggin (2019 a), where constraints on asset holdings are imposed exogenously. However, an analysis in terms of bounded awareness allows for sharper results. Most importantly, we derive the following result: Consider two agents where j has coarser awareness than i, and suppose that they have the same discount factor, and that their beliefs with respect to j's partition are the same. Then i has a lower survival index than j.

It may then be shown that, with nested awareness and consistent beliefs, all agents survive. However, there are equilibria in which j survives but i does not. That is, markets might select for lower levels of awareness, when a higher level of awareness means that the agent also has beliefs further from the truth.

These results raise important questions about welfare. Intuitively, since coarse awareness precludes some trades, we would expect it to be welfare reducing. We give conditions under which insurance against idiosyncratic risk is incomplete. Differential awareness can thus lead to a restriction of mutually beneficial trades and in extreme cases, completely eliminate any trade.

Nevertheless, as the analysis of survival shows, there are conditions under which bounded awareness may protect agents from making costly mistakes. We use the concepts of No Betting Pareto improvements due to Gilboa, Samuelson and Schmeidler (2014), henceforth GSS (2014) and true-Pareto-efficieny by Blume et al. (2018) to formalize this idea.

Next, we examine the options available to agents who understand that they may have bounded awareness, but cannot incorporate this understanding in the state-act model in which awareness may be represented formally. Such agents may constrain their choices using heuristics which are, in the terminology of Gigerenzer (2007), ecologically rational.

Finally, we offer some concluding comments and directions for future research.

2 The Model

The 'true' model of the economy is the same as that of Guerdjikova and Quiggin (2019 a) and will be restated briefly.

2.1 The "True" Model of the Economy

Let $\mathbb{N} = \{0; 1; 2; ..\}$ denote the set of time periods. Uncertainty is modelled through a sequence of random variables $\{\mathcal{S}_t\}_{t\in\mathbb{N}}$ each of which takes values in a finite set S, with $S_0 = \{s_0\}$. Events (subsets of S) are denoted ω. Denote by $s_t \in S$ the realization of random variable $\mathcal{S}_t$. Denote by $\Omega = \prod_{t\in\mathbb{N}} S$ the set of all possible observation paths, with representative element $\sigma = (s_0; s_1; s_2 \ldots s_t \ldots)$. Finally denote by $\Omega_t = \prod_{\tau=0}^{t} S$ the collection of all finite paths of length t, with representative element $\sigma_t = (s_0; s_1; s_2 \ldots s_t)$. Define the cylinder with base on $\sigma_t \in \Omega_t$, $t \in \mathbb{N}$ as $Z(\sigma_t) = \{\sigma \in \Omega | \sigma = (\sigma_t \ldots)\}$. Let $\mathbb{F}_t = \{Z(\sigma_t) : \sigma_t \in \Omega_t\}$ be a partition of the set Ω. Clearly, $\mathbb{F} = (\mathbb{F}_0 \ldots \mathbb{F}_t \ldots)$ denotes a sequence of finite partitions of Ω such that $\mathbb{F}_0 = \Omega$ and $\mathbb{F}_t$ is finer than $\mathbb{F}_{t-1}$.Let $\mathcal{F}_t$ be the σ-algebra generated by partition $\mathbb{F}_t$, and let $\mathcal{F}$ be the σ-algebra generated by $\cup_{t\in\mathbb{N}}\mathcal{F}_t$. It can be shown that $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is a filtration. We define on $(\Omega; \mathcal{F})$ a probability distribution π. We will assume that the true process of the economy is i.i.d. and write $\pi(s_{t+1} = s \mid \sigma_t) =: \pi(s)$.

There is a single good consumed in positive quantities. There is a finite set I with $|I| = n$ of infinitely lived agents. Each agent's welfare depends on their consumption stream $c^i : \Omega \to \prod_{t\in\mathbb{N}}\mathbb{R}_+$. Each agent i is endowed with a consumption plan, denoted e^i. The total endowment of the economy is denoted by $e = \sum_i e^i$.

Agents are assumed to be expected utility maximizers given their knowledge about the economy and their (subjective) beliefs[2]. Agent i's utility function for risk is denoted by u_i and his discount factor is β^i. Since we are not concerned with differences in time preference, we will simplify by assuming that all discount factors are the same $\beta^i = \beta < 1$.

We will impose the following assumptions on utility functions and endowments, which are standard in the survival literature:

Assumption 1 All agents are expected utility maximizers with utility functions for risk $u_i : \mathbb{R}_+ \to \mathbb{R}$ which are twice continuously differentiable, strictly concave, and satisfy $\lim_{c\to 0} u_i'(c) = \infty$ and $\lim_{c\to\infty} u_i'(c) = 0$.

Assumption 2 Individual endowments are strictly positive, $e^i(\sigma_t) > 0$ for all i and σ_t. Aggregate endowments are uniformly bounded away from zero and uniformly bounded from above. Formally, there is an $m > 0$ such that $\sum_{i\in I} e^i(\sigma_t) > m$ for all i, σ_t; moreover, there is an $m' > m > 0$ such that $\sum_{i\in I} e^i(\sigma_t) < m'$ for all σ_t.

2.2 Modelling Unawareness as Coarsening

In this paper, we think of unawareness as the inability of the agent to form a sufficiently fine perception of the state space. A partially aware agent i will perceive a state space Ω^i coarser than S, in which some states with potentially different consumption allocations are coalesced into a single perceived state. To understand the process, it is helpful to think in syntactic (propositional terms). Each state in S may be described in terms of the truth values of a set of propositions P describing relevant contingencies, in this case, related to endowments.

An agent may be less aware than another because the set of descriptions available to them is coarser. For example, a relatively unaware agent might consider the proposition 'the economy is (or is not) at full employment', giving rise to a state space with two elements. A more aware agent might distinguish the various phases of the economic cycle, such as 'peak', 'contraction', 'trough' and 'expansion'. An even more aware agent might consider a state space in which the states were indexed by the rate of growth of gross domestic product.

An alternative form of coarsening arises when some agents display 'pure unawareness' of relevant propositions (Li 2009). For example, two agents might have access to the same set of propositions to describe the state of the domestic economy, but only one of them might consider developments in the world economy. The more aware agent would have access to a state space derived as the Cartesian product of the state of the domestic economy and the state of

[2] An expected utility representation with a coarse subjective state space has been recently axiomatized by Minardi and Savochkin (2016).

the world economy, while the less aware agent would have access to a coarser quotient space, in which all states of the world economy were treated as indistinguishable. We would expect the less aware agent to display 'home bias' (French and Poterba 1991).

We now formalize the idea that some agents perceive a coarser state space than the one given by S. In particular, agent i is assumed to be aware of a partition of S given by $W^i = \left\{w_1^i \ldots w_{K_i}^i\right\}$, where each $w_k^i \subseteq S$, $w_k^i \cap w_{k'}^i = \varnothing$ for any $k \neq k'$ and $\cup_{k=1}^{K_i} w_k^i = S$. This is a specific type of unawareness: the agent's perception of the world is coarser than reality in that he cannot distinguish between those states which are grouped in a given w_k^i.

We assume that all fully aware agents have identical information and that the information revelation process for them is represented by the sequence $\mathbb{F}$. A fully aware agent can distinguish any two nodes σ_t and σ_t'. By contrast, a partially aware agent cannot distinguish nodes σ_t and σ_t' if and only if, for every $\tau \leq t$, s_τ, $s_\tau' \in w_{k_\tau}^i$ for some $w_{k_\tau} \in W^i$. Hence, for a partially aware agent, the paths he is aware of can be written as $\Omega^i = \prod_{t \in \mathbb{N}} W^i$ with a representative element $\omega^i = \left(w_0 = \{s_0\}; w_1^i \ldots w_t^i \ldots\right)$. Denote by Ω_t^i the set of paths of length t.

From the point of view of agent i, the information revelation is described by finite partitions of the set Ω^i, $\left(\mathbb{F}_t^i\right)_{t \in \mathbb{N}}$ defined in analogy to $\left(\mathbb{F}_t\right)_{t \in \mathbb{N}}$. Note that for each t, $\mathbb{F}_t^i$ is coarser than the corresponding $\mathbb{F}_t$. We will denote by $\mathcal{F}_t^i$ the σ-algebra generated by partition $\mathbb{F}_t^i$. $\mathcal{F}_0^i = \mathcal{F}_0$ is the trivial σ-algebra. Let $\mathcal{F}^i$ be the σ-algebra generated by $\cup_{t \in \mathbb{N}} \mathcal{F}_t^i$. Just as above, $\left\{\mathcal{F}_t^i\right\}_{t \in \mathbb{N}}$ is a filtration.

Agent i's beliefs π^i are defined on $\left(\Omega^i; \mathcal{F}^i\right)$. The one-step ahead probability distribution $\pi^i\left(w_{t+1}^i \mid \omega_t^i\right)$ is defined analogously to $\pi\left(s_{t+1} \mid \sigma_t\right)$.

Obviously, $\mathcal{F}$ is finer than $\mathcal{F}^i$ and hence, the true probability distribution π on $(\Omega; \mathcal{F})$ specifies a probability distribution on $\left(\Omega^i; \mathcal{F}^i\right)$ with

$$\pi\left(\omega_t^i = \left(w_0 \ldots w_t^i\right)\right) = \pi\left\{\sigma_t \mid s_\tau \in w_\tau^i \text{ for all } \tau \in \{1 \ldots t\}\right\}.$$

We will say that i's beliefs are correct if they coincide with the restriction of π to $\left(\Omega^i; \mathcal{F}^i\right)$.

For most of the paper, we will restrict attention to beliefs which describe an i.i.d. process, $\pi^i\left(w_{t+1}^i = w^i \mid \omega_t^i\right) = \pi^i\left(w^i\right)$. We next assume that all states are non-null under the true one-step-ahead probability and all agents have one-step-ahead beliefs that are absolutely continuous with respect to the truth.

Assumption 3 $\pi(s) > 0$ for all $s \in S$ and for all $i \in I$, $\pi^i\left(w^i\right) > 0$ for all $w^i \in W^i$.

Since in general, $\pi^i\left(w^i\right) \neq \pi\left(w^i\right)$, we introduce the Kullback-Leibler (K-L) distance as a measure of deviation of agents' beliefs from the truth.

Definition 1 *For a given partition of S, W^i, the Kullback-Leibler (K-L) distance of agent i's beliefs π^i with respect to the truth π is given by:*

$$\sum_{w^i \in W^i} \pi\left(w^i\right) \ln \frac{\pi\left(w^i\right)}{\pi^i\left(w^i\right)}.$$

As usual, when i's beliefs on W^i are correct, the K-L distance is 0, whereas the violation of the absolute-continuity property posited in Assumption 3 would lead to a K-L distance of infinity.

We will require that each agent is aware of their consumption stream and, in particular, of their initial endowment:

Assumption 4 The consumption set of i consists of functions $c^i : \Omega \to \prod_{t \in \mathbb{N}} \mathbb{R}_+$ measurable with respect to $\left(\Omega^i; \mathcal{F}^i\right)$. In particular, the initial endowment of agent i, e^i is $\mathcal{F}^i$-measurable.

Under differential, but constant awareness, this measurability constraint will be the defining feature of the economy. In particular, the equilibrium under differential awareness will differ from the standard notion of equilibrium in that we require both the initial endowments and the final consumption of each agent to be measurable with respect to this agent's awareness partition.

In general, we can thus distinguish three types of processes depending on the space on which they are measurable. The universal state space Ω describes all possibly relevant contingencies in the economy and provides the most detailed description of the world. The set of processes (consumption, endowment, asset payoffs, prices) measurable with respect to $(\Omega, \mathcal{F})$ thus allows the outcome to depend on any such contingency. This is the richest set of processes feasible for the economy at hand.

In reality, agents face two types of restrictions. The first relates to their awareness and is captured by an awareness partition Ω^i which is a coarsening of Ω. In general, Ω^i provides a less detailed description of the world. An agent with awareness Ω^i can only envision processes which are measurable with respect to $\left(\Omega^i, \mathcal{F}^i\right)$. For each process measurable with respect to $\left(\Omega^i, \mathcal{F}^i\right)$, there exists a unique process measurable with respect to $(\Omega, \mathcal{F})$ which specifies identical outcomes on all $\omega_t \in \omega_t^i$ for any $\omega_t^i \in \Omega_t^i$ and for each t. Thus, the set of processes perceived by agent i are isomorphic to a subset of those potentially relevant for the economy. The converse is however not true. Some processes relevant for the economy cannot be described in terms of $\left(\Omega^i, \mathcal{F}^i\right)$. The agent, being unaware of the more detailed description of states, will be unable to conceive of such non-measurable processes. This in turn, restricts his trades in that his initial endowment, his asset holdings as well as his final consumption have to be measurable with respect to his awareness partition $\left(\Omega^i, \mathcal{F}^i\right)$.

The second type of restriction is related to the tradeability of consumption claims on potentially relevant contingencies. The common assumption of market completeness is equivalent to the existence for each state ω_t^i of an Arrow security which pays one unit on this state and zero otherwise. Realistically, however, not all such trades are possible and only a subset of the necessary assets are available in the economy. Instead of spanning the entire space Ω, the set of available assets might only span a partition of it, Ω^A. We are specifically interested in the case, in which this partition can be derived from a partition W^A on the one-period-ahead state-space. In Guerdjikova and Quiggin (2019 b) we show, using results from Kountzakis and Polyrakis (2006), that under certain conditions, an incomplete set of assets A will indeed span such a partition with the corresponding state space $\left(\Omega^A, \mathcal{F}^A\right)$. In particular, the set of available assets is identical to a set of generalized Arrow securities, which pay 1 only if a particular element w^A of the partition W^A was realized at some node, that is, $\omega_t^A = \left(\omega_{t-1}^A, w^A\right)$ and 0, otherwise. Similarly to the constraints imposed by unawareness, incomplete markets impose constraints on the trades that can be executed in markets.

The two types of constraints are connected. Existing assets in the economy have to be describable in the agents' language and thus have to specify payoff streams measurable with respect to at least one agent's awareness. Conversely, agents' understanding of the economy might be enhanced by the existence of assets that pay conditional on certain events. For example, the existence of insurance against earthquakes might make some agents aware of the possibility of an earthquake in the area.

For the purposes of this paper, we will thus assume that the partition induced by the set of assets in the economy, Ω^A is identical to the coarsest common refinement of the individual partitions, Ω^i. Hence, each agent can trade in all Arrow securities which pay on the elements of his partition Ω^i. That is, from the point of view of each agent, markets are complete. If the agents' awareness partitions are nested, the partition of the most aware agent coincides with the market partition and the most aware agent "understands" all available asset payoffs.

When agents' partitions are not nested, the coarsest common refinement of $\left(\Omega^i\right)_i$ will contain elements which are not expressible in any of the agents' awareness. The restrictions on trade imposed by the agents' awareness, however, will guarantee that each agent holds a portfolio, the payoff of which is measurable with respect to his awareness.

As usual, the equilibrium can be obtained in two ways. One possibility is to consider a pure exchange economy. Agents can trade their initial endowment for a new state-contingent consumption stream which satisfies the measurability constraints. In particular, each agent can sell his initial endowment to a market maker at the announced market prices and buy in return a consumption bundle that is optimal given market prices and his awareness level.

A second possibility is to endow agents with initial portfolios of assets which pay conditional on the state realization such that the payoffs are measurable with respect to their respective awareness partitions. Agents can sell their portfolios to a market maker at the announced market prices and buy in return a portfolio that is optimal given market prices and the agent's awareness level.

If the asset structure of the economy satisfies the conditions introduced above, the two market mechanisms lead to identical equilibrium state prices and consumption streams.

3 Equilibrium in Markets with Differential Awareness

Our main results are derived on the assumption that agents trade their endowments at time 0 with no subsequent opportunity for retrading. Intuitively, this corresponds to the case, in which the agents' awareness remains unchanged and they do not respond to price changes which are conditional on events they are unaware of. Thus, the approach taken in the main part of the paper mimics that of Sandroni (2005), in which there is a single period of trade, but information is subsequently revealed according to the structure presented in Section 3. This assumption greatly simplifies the analysis and allows us to derive a simple criterion for survival in economies with differential awareness. Differently from Sandroni, we allow consumption to occur in time. In Appendix A of Guerdjikova and Quiggin (2019 a), we show how the analysis can be extended to the case of constant awareness with sequential trading. Even though the definition and the analysis of the equilibrium are substantially different for the two cases, we show that the main insights of the paper are robust to such a modification.

Definition 2 *An equilibrium of the economy with differential awareness consists of an integrable*[3] *price system* $(p(\sigma_t))_{\sigma_t \in \Omega}$ *and a consumption stream* c^i *for every agent* i *such that (i) all agents* $i \in I$ *are maximizing their expected utility given the price system subject to choosing consumption streams measurable with respect to their awareness partition; and (ii) markets clear:*

$$c^i = \arg\max_{c^i} V_0^i\left(c^i\right) = \arg\max_{c^i} \left\{ \begin{array}{c} u_i\left(c^i\left(\sigma_0\right)\right) + \sum_{t=1}^{\infty} \beta_i^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i\left(\omega_t^i\right) u_i\left(c^i\left(\omega_t^i\right)\right) \\ s.t. \sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} \sum_{\sigma_t \in \omega_t^i} p\left(\sigma_t\right) c^i\left(\omega_t^i\right) \\ \leq \sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} \sum_{\sigma_t \in \omega_t^i} p\left(\sigma_t\right) e^i\left(\omega_t^i\right) \end{array} \right\} \quad (1)$$

$$\sum_{i \in I} c^i\left(\sigma_t\right) = \sum_{i \in I} e^i\left(\sigma_t\right) \ \forall \sigma_t \in \Omega$$

An equilibrium in an economy with differential awareness is consistent with the fact that different agents trade on different partitions of the state space and, hence, effectively optimize over different sets of commodities (consumption on events ω_t^i, rather than σ_t). The equilibrium can be interpreted in the following way: in period 0, before any uncertainty is resolved, all agents sell their initial endowment to an intermediary[4] at market prices and use the revenues to buy their preferred consumption streams c^i for all future contingencies of which

[3] Integrability of $(p(\sigma_t))_{\sigma_t \in \Omega}$, on $(\Omega; \mathcal{F}; \mu)$, where μ is the counting measure, or equivalently, the requirement that the price system is L^1 on $(\Omega; \mathcal{F}; \mu)$, ensures that the total wealth of an individual agent is finite, i.e., that the sum $\sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} \sum_{\sigma_t \in \omega_t^i} p(\sigma_t) e^i\left(\omega_t^i\right)$ is well-defined, see Bewley (1972, p. 516).

[4] The fact that agents can trade through an intermediary means that the restriction of measurability is imposed only on the total net trades of a given agent. One could alternatively define an equilibrium through bilateral trades and require that the bilateral net trades be measurable with respect to each agent's partition. This will in general restrict the set of potential equilibrium allocations. Note, however, that when agents' partitions are nested,

they are aware. The price of consumption contingent on a coarse contingency ω_t^i is simply the sum of consumption prices over all nodes $\sigma_t \in \omega_t^i$, that is, $\sum_{\sigma_t \in \omega_t^i} p(\sigma_t)$.

Proposition 1 in Guerdjikova and Quiggin (2019 a, p. 1704) implies that under Assumptions 1–3, an equilibrium of the economy with differential unawareness exists. Furthermore, the equilibrium satisfies: for each $i \in I$ and at each ω_t^i, $\omega_{t+1}^i \in \Omega^i$ such that $\pi\left(\omega_{t+1}^i\right) > 0$,

$$\frac{u_i'\left(c^i\left(\omega_t^i\right)\right)}{\beta_i \pi^i\left(\omega_{t+1}^i \mid \omega_t^i\right) u_i'\left(c^i\left(\omega_{t+1}^i\right)\right)} = \frac{p\left(\omega_t^i\right)}{p\left(\omega_{t+1}^i\right)} = \frac{\sum_{\sigma_t \in \omega_t^i} p(\sigma_t)}{\sum_{\sigma_{t+1} \in \omega_{t+1}^i} p(\sigma_{t+1})}, \tag{2}$$

where $p(\cdot)$ is the equilibrium price system.

The following propositions aim to illustrate the way in which differential awareness affects equilibrium allocations. The first result considers an economy, in which some agent(s) cannot distinguish between two states (σ_t' and σ_t'') in which aggregate endowment differs, but do(es) differentiate between two states (σ_t and σ_t') in which the aggregate endowment is the same. Such a situation can arise when the agent's own endowment is constant across σ_t' and σ_t'', but not across σ_t and σ_t'. The requirement that the agent have the same equilibrium consumption in σ_t' and σ_t'' then has two effects: first, it prevents the economy from obtaining full insurance against idiosyncratic risk (at the cost of providing insurance against aggregate risk to the agent in question) and second, it leads to prices being biased as compared to the true probabilities of the states, even when all agents hold correct beliefs.

Proposition 3 *Consider an economy with differential awareness and correct beliefs. Suppose that for some σ_t, σ_t', $\sigma_t'' \in \Sigma$ $e(\sigma_t) = e(\sigma_t') \neq e(\sigma_t'')$. Let furthermore, for some $i \in I$, and some $\omega_t^i \in \Omega^i$, σ_t', $\sigma_t'' \in \omega_t^i$. Finally, assume that there are distinct ω_t, ω_t' and ω_t'' with $\sigma_t \in \omega_t$, $\sigma_t' \in \omega_t'$ and $\omega_t'' = \omega_t^i \backslash \omega_t'$ such that for any $j \in I$, $\omega_t \in \Omega_t^j$ and either ω_t', $\omega_t'' \in \Omega_t^j$ or $\omega_t^i \in \Omega_t^j$. Then:*

(i) the equilibrium of the economy with differential awareness provides insurance to i against the aggregate risk on σ_t' and σ_t'';

(ii) the equilibrium of the economy with differential awareness does not provide full insurance against idiosyncratic risk;

(iii) the price ratio $\frac{p(\omega_t)}{p\left(\omega_t^i\right)}$ is biased relative to the probabilities of ω_t and ω_t^i.

Proposition 4 *Suppose that there are states σ_t, σ_t' and $\sigma_t'' \in \Sigma$ such that $e(\sigma_t) = e(\sigma_t') \neq e(\sigma_t'')$ and two sets of agents $\mathcal{I}$ and $\mathcal{J}$ with $\mathcal{I} \cup \mathcal{J} = I$ with*

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the agent with the finest partition can de facto play the role of an intermediary and thus, measurable bilateral net trades supporting the equilibrium allocation always exist. The same is true for an economy with two agents with non-nested partitions. More generally, the two equilibrium notions will not coincide and this might have an impact on the existence and the properties of the equilibrium, as well as on survival results. Note, however, that in the cases studied below, notably nested partitions, or an economy with a fully aware agent, the results on survival will not depend on the definition chosen.

$\omega_t^i = \{\sigma_t, \sigma_t''\} \in \Omega_t^i$ *for all* $i \in \mathcal{I}$ *and* $\omega_t^j = \{\sigma_t', \sigma_t''\} \in \Omega_t^j$ *for all* $j \in \mathcal{J}$*. Then,*

$$\sum_{j \in \mathcal{J}} \left[e^j(\sigma_t) - c^j(\sigma_t)\right] = \sum_{j \in \mathcal{J}} \left[e^j(\sigma_t') - c^j(\sigma_t')\right]$$

that is, agents in $\mathcal{J}$ *and in* $\mathcal{I}$ *cannot mutually insure each other against the idiosyncratic risk between* σ_t *and* σ_t'*.*

The result of Proposition 4 is of particular interest in the case in which agents in $\mathcal{I}$ are consistently poorer than those in $\mathcal{J}$ in σ_t and consistently richer than those in $\mathcal{J}$ in σ_t'. A standard equilibrium would optimally predict that agents in $\mathcal{I}$ and in $\mathcal{J}$ would mutually insure each other against the idiosyncratic risk in states σ_t and σ_t'. Yet, when the partitions of the agents in $\mathcal{I}$ and $\mathcal{J}$ intersect in σ_t'', a state with a total initial endowment distinct from both σ_t and σ_t', a positive transfer from $\mathcal{J}$ to $\mathcal{I}$ in σ_t implies also a positive transfer from $\mathcal{J}$ to $\mathcal{I}$ in σ_t', implying that such mutual insurance is impossible.

The proposition shows a sense in which trade is limited when awareness partitions are non-nested. Differential awareness can thus lead to a restriction of mutually beneficial trades and in extreme cases, completely eliminate any trade.

Our last result in this section provides conditions that allow us to "observationally distinguish" between a standard economy and an economy with differential awareness.

Proposition 5 *Consider an economy with differential awareness. If for some* $i \in I$ *and some* σ_t, $\sigma_t' \in \Sigma$ *with* $e(\sigma_t) > e(\sigma_t')$, σ_t, $\sigma_t' \in \omega_t^i$ *for some* $\omega_t^i \in \Omega^i$*, then*

(*i*) *there is no economy with full awareness and homogenous beliefs satisfying Assumptions 1 and 3 and initial endowment process e such that its equilibrium coincides with the equilibrium of the economy with differential awareness;*

(*ii*) *if an economy with full awareness and i.i.d. beliefs* $\left(\tilde{\pi}^k\right)_{k \in I}$ *satisfying Assumptions 1 and 3, and with an initial endowment process e has an equilibrium that coincides with the equilibrium of the economy with differential awareness, then there is an agent* j *such that* $\frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma_t')} < \frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma_t')}$*, that is,* i *underestimates the probability of the "good" state of the economy* σ_t *relative to* j*.*

If furthermore, there are s, $s' \in S$ *such that* $\sigma_t = (\sigma_{t-1}, s)$, $\sigma_t' = (\sigma_{t-1}, s')$, s, $s' \in w^i$ *for some* $w^i \in W^i$ *and if for some* $\sigma_{t'}' \in \Sigma$, $e(\sigma_{t'}', s) < e(\sigma_{t'}', s')$*, then*

(*iii*) *there is no economy with full awareness and i.i.d. beliefs satisfying Assumptions 1 and 3 and initial endowment process e such that its equilibrium coincides with the equilibrium of the economy with differential awareness.*

Proposition 5 illustrates the differences between "standard" economies and economies with differential awareness. Agents with lower awareness levels will appear to be underestimating the probability of "good" states. As a consequence, an economy with differential awareness will, in general, exhibit the equity premium puzzle, overpricing bonds. More generally, assets that are measurable with respect to partitions of agents with lower awareness levels and thus do not expose such agents to surprises, will be overpriced. Furthermore, when the economy is known to be i.i.d., but endowment reversals across states can occur over time, the behavior of partially aware agents cannot be explained by i.i.d. beliefs, since the state that the agent overweighs will change depending on the endowment of the economy. While this behavior is reminiscent of ambiguity aversion, we will see below that the long-run behavior of the economy and in particular, the implications for survival, are very different from that of an economy with ambiguity-averse agents. Indeed, while Condie (2008) shows that agents with max-min preferences a.s. vanish in the presence of expected utility maximizers with correct beliefs, our results below demonstrate that boundedly aware agents can survive and affect prices in the long-run.

4 Survival in Economies with Coarse Contingencies

In the previous sections, we showed that differential awareness can have an impact on equilibrium prices and allocations. This raises the question of whether the impact of less aware agents on prices and allocations is temporary or permanent. Is it the case that their consumption converges to 0 over time, thus driving the equilibrium allocation to the one that would have obtained had all agents been fully aware? In this section, we will show that partially aware agents can have a long-term impact on prices and risk sharing.

We define survival as usual:

Definition 6 *Agent i vanishes on a path σ if $\lim_{t\to\infty} c^i(\sigma_t) = 0$. Agent i survives on σ if $\lim_{t\to\infty} \sup c^i(\sigma_t) > 0$.*

If Assumptions 1-4 hold, the results we derive in Guerdjikova and Quiggin (2019 a) can be adapted to and summarized in the context of differential awareness as follows[5]. We recall that discount factors are equal across agents, so survival will depend only on agents' awareness structure and beliefs. The first remark explains why aggregate uncertainty is necessary in order for differential awareness to have an effect:

Remark 7 *In an economy with no aggregate uncertainty and identical correct beliefs, all agents will be fully insured. Indeed, since a full insurance consumption stream is measurable with respect to any individual partition, the measurability*

[5] The proofs to the remarks in this section are contained in Guerdjikova and Quiggin (2019 a).

conditions in such an economy are not binding in equilibrium. Hence, all agents will survive regardless of their individual awareness partition. In this case, the first-order conditions (2) (with correct beliefs) and the equilibrium allocation coincide with those in an economy with full awareness.

Our first set of results concerns the case of agents with "nested" awareness partitions, that is, the case when agents in the economy can be ordered with respect to their awareness from "most" to "least" aware. To state the results, we provide two definitions that give conditions for an agent's unawareness to be relevant in the limit with respect to the endowment of the economy as a whole or with respect to the unawareness of a different agent.

Definition 8 *The unawareness of agent i, given by the partition Ω^i, is irrelevant in the limit if for any $\omega^i \in \Omega^i$ and any σ, $\sigma' \in \omega^i$, $\lim_{t\to\infty} [e(\sigma_t) - e(\sigma'_t)] = 0$. The unawareness of agent i, given by the partition Ω^i, is relevant in the limit if for some $w^i \in W^i$, s and $s' \in w^i$, there is an $\epsilon > 0$ such that for any σ, $\sigma' \in \omega^i$,*

$$\lim_{t\to\infty} \sup [e(\sigma_t; s) - e(\sigma'_t; s')] > \epsilon. \tag{3}$$

The unawareness of agent i is considered irrelevant if, in the limit, the total endowment of the economy is measurable with respect to agent i's partition. Such an agent is effectively aware of and thus can trade on the total endowment process of the economy in the limit. In contrast, agent i's unawareness is relevant even in the limit, if there are at least two states that i cannot distinguish and in which the total endowment of the economy remains distinct.

Note that if i's unawareness is irrelevant in the limit, then so are those of any agent j who is more aware and has a partition Ω^j finer than Ω^i. Similarly if i's awareness is relevant in the limit, then so is that of a less aware agent j with a partition Ω^j coarser than Ω^i.

Consider agent j and for any $\omega^j \in \Omega^j$ with $\omega^i \subseteq \omega^j$, define the set $\hat{\Omega}^i_t\left(\omega^j_t\right) = \{\omega^i_t \in \Omega^i_t \mid \omega^i_t \subseteq \omega^j_t \text{ s.t. } \min_{\sigma_t\in\omega^j_t} e(\sigma_t) = \min_{\sigma_t\in\omega^i_t} e(\sigma_t)\}$, the set of ω^i_t on which the initial endowment of the economy obtains its minimum with respect to the set ω^j_t. Let $\check{\Omega}^i_t\left(\omega^j_t\right) = \left\{\omega^i_t \subseteq \omega^j_t\right\} \setminus \hat{\Omega}^i_t\left(\omega^j_t\right)$.

Definition 9 *Let the awareness partition of agent i, Ω^i be finer than that of agent j, Ω^j. The unawareness of agent j given by the partition Ω^j, is irrelevant in the limit with respect to that of agent i given by partition Ω^i if for any $\omega^i \in \Omega^i$ and $\omega^j \in \Omega^j$ s.t. $\omega^i \subseteq \omega^j$, $\lim_{t\to\infty} \check{\Omega}^i_t\left(\omega^j_t\right) = \varnothing$. The unawareness of agent j is relevant in the limit with respect to that of agent i if there is an $\epsilon > 0$, $w^i \in W^i$ and $w^j \in W^j$, $w^i \subseteq w^j$ such that for any $\omega^j \in \Omega^j$ and every $\omega^i \subseteq \omega^j$, $\omega^i \in \Omega^i$,*

(i) $\min_{(\sigma_{t^k};s)\in\left(\omega^i_{t^k};w^i\right)} e(\sigma_{t^k}; s) - \min_{(\sigma_{t^k};s)\in\left(\omega^j_{t^k};w^j\right)} e(\sigma_{t^k}; s) > \epsilon$ occurs on an infinite set of periods $\left(t^k\right)_k$ such that

(ii) $\min_{\left\{\sigma_{t^k+1}\in\omega^i_{t^k+1} \mid \omega^i_{t^k+1}\in\check{\Omega}^i_{t^k+1}\left(\omega^j_{t^k};w^j\right)\right\}} e(\sigma_{t^k+1}) - \min_{(\sigma_{t^k};s)\in\left(\omega^j_{t^k};w^j\right)} e(\sigma_{t^k}; s) > \epsilon$ for all t^k.

To understand the definition, note that, in general, the initial endowment of the economy is not measurable with respect to to Ω^i or Ω^j. The maximum consumption that j is aware of at ω_t^j, given the initial endowment of the economy, is $\min_{\sigma_t \in \omega_t^j} e(\sigma_t)$, whereas the maximum consumption that i is aware of at $\omega_t^i \subseteq \omega_t^i$ is $\min_{\sigma_t \in \omega_t^i} e(\sigma_t)$. Furthermore, if $\check{\Omega}_t^i\left(\omega_t^j\right) = \varnothing$, then these two values coincide for all $\omega_t^i \subseteq \omega_t^j$: even though j's partition is coarser, his awareness about the maximal possible consumption of the economy is the same as that of i on ω_t^j. If this property obtains in the limit, we say that j's unawareness is irrelevant in the limit with respect to that of i. If, in contrast, $\check{\Omega}_t^i\left(\omega_t^j\right) \neq \varnothing$, then i is aware that he can obtain a strictly higher consumption on ω_t^i than j on ω_t^j, that is, j's unawareness is "relevant" with respect to that of i. The condition for j's constraints to be relevant with respect to those of i in the limit requires that (i) on every path $\omega^i \subseteq \omega^j$, on which w^i occurs infinitely often (i.o.) the maximal consumption of which i is aware exceeds that of which j is aware by ϵ i.o. and (ii) on every path ω^j, on which w^j occurs i.o. the minimal non-zero difference in maximal consumption of which i and j respectively are aware on ω^j exceeds ϵ i.o..

Remark 10 *Consider an economy with differential awareness:*

(i) *if the agents in the economy have nested awareness partitions and correct beliefs, all agents survive a.s.;*

(ii) *if agent i has a (weakly) finer awareness partition Ω^i than agent j, Ω^j and if for the partition W^j, the K-L distance of agent's i beliefs from the truth is strictly smaller than that of j, j vanishes a.s.;*

(iii) *if the agents in the economy have nested awareness partitions, Ω^1 strictly finer than Ω^2... strictly finer than Ω^n and identical beliefs, and if the unawareness of any agent $i \geq 2$ is both relevant in the limit and relevant with respect to that of $i-1$, all agents survive a.s.;*

(iv) *if the agents in the economy have nested awareness partitions, Ω^1 strictly finer than Ω^2... strictly finer than Ω^n, if for all $i < j$,*

$$\sum_{w^j} \pi\left(w^j\right) \ln \frac{\pi\left(w^j\right)}{\pi^i\left(w^j\right)} > \sum_{w^j} \pi\left(w^j\right) \ln \frac{\pi\left(w^j\right)}{\pi^j\left(w^j\right)}$$

and if the unawareness of any agent $i \geq 2$ is both relevant in the limit and relevant with respect to that of $i-1$, agents 1 and 2 a.s. survive. If, in addition[6]*, for every $j \in \{2...n-1\}$, all $w^{j+1} \in W^{j+1}$ and all $w^j \subseteq w^{j+1}$, $\pi^j\left(w^j \mid w^{j+1}\right) = \pi\left(w^j \mid w^{j+1}\right)$, all agents a.s. survive.*

Our first result (i) shows that whenever agents have correct beliefs relative to their awareness partitions and the awareness partitions are ordered with respect to inclusion, their level of awareness is irrelevant for survival. In fact, all agents survive. We can relate this result to the features identified in Propositions 3

[6]Since $\pi^j\left(w^{j+1}\right)$ is in general incorrect, this does not imply that $\pi^j\left(w^j\right)$ is correct.

and 5: recall that in economies with differential awareness, insurance against idiosyncratic risk did not obtain in equilibrium. In contrast, more aware agents would insure less aware agents against some of the aggregate risk. Finally, relative state prices would be biased relative to the state probabilities even if all agents hold correct beliefs. The result in part (i) of Remark 10 implies that these features of the economy will persist in the long run, even if all agents have correct beliefs and equal discount factors. Part (iii) extends this result to the case of identical, but not necessarily correct beliefs.

The second insight concerns agents with wrong beliefs. When agents are simultaneously less aware and hold beliefs further from the truth than others, they almost surely vanish, as shown in part (ii). However, more awareness can compensate for wrong beliefs. Part (iv) considers the case in which agents have nested partitions such that the less aware agents have beliefs closer to the truth. It requires that for $i \geq 2$, agents' constraints are relevant even in the limit. In such a scenario, the less aware agents cannot consume the entire endowment of the economy: such a consumption stream would violate the constraint that their consumption be measurable with respect to their awareness partition. Hence, it is the agents with beliefs further away from the truth, but with higher levels of awareness, who ensure that the markets clear. They consume the 'leftovers' of the less aware agents and, thus, the fact that the latter's unawareness is relevant ensures that they survive a.s..

To contrast survival in economies with differential awareness to that in an economy with full awareness, consider an extreme version of the Blume and Easley (2006) model with a continuum of agents. If all beliefs have positive support, then only agents with perfectly accurate beliefs survive. More generally, only agents with maximally accurate beliefs (that, is minimal K-L distance from the truth) can survive. By contrast, with differential awareness and nested partitions, agents with different beliefs, varying in accuracy, can coexist. This results seems more consistent with observed outcomes.

We next consider economies, in which agents' awareness partitions are not nested. We start with a formal definition of economies with non-nested awareness partitions.

Definition 11 *Agents i and j have non-nested awareness partitions if there are states*[7] s*,* s'*,* s''*,* $s''' \in S$ *such that:*

- *there are elements of i's awareness partition w^i, $w^{i\prime}$, $w^{i\prime\prime} \in W^i$ with $w^i \neq w^{i\prime}$ such that $s \in w^i$, $s' \in w^{i\prime}$ and s'', $s''' \in w^{i\prime\prime}$ and*

[7] The definition does not require the four states to be distinct and thus also applies to economies with only 3 states, where one can set $s' = s''$. However, requiring $s' = s''$ is in general too restrictive for our purposes, since it excludes, for example, an economy in which $W^i = \{\{s\}; \{s'\}; \{s''; s'''\}\}$ and $W^j = \{\{s; s'\}; \{s''\}; \{s'''\}\}$. Indeed, choose any three states (for example, s, s'' and s''') and note that at least one of the agents (here: j) can distinguish among any of the three states and hence, the definition of nonnested partitions would not apply, contrary to intuition. In economies with only two states, agents' partitions are trivially nested.

- *there are elements of j's awareness partition w^j, $w^{j''}$, $w^{j'''} \in W^i$ with $w^{j''} \neq w^{j'''}$ such that s, $s' \in w^j$, $s'' \in w^{j'}$ and $s''' \in w^{j'''}$.*

If the states s and s' satisfy this definition, then we will say that i can distinguish s and s' and trade between them, whereas j cannot.

We will say that agents in the economy have non-nested awareness partitions if, for each agent i, there are states s and $s' \in S$ between which i can distinguish and trade, but between which no other agent in the economy can distinguish, nor trade.

Remark 12 *Consider an economy with differential awareness and assume that for an agent j, there are states $s(j)$ and $s'(j) \in S$ such that the awareness partitions of j and any other agent $k \in I \backslash \{j\}$ are non-nested and j can distinguish and trade between $s(j)$ and $s'(j)$, whereas k cannot. Assume that condition (3) holds for $s(j)$ and $s'(j)$.*

(i) Agent j survives a.s..

(ii) If, furthermore, the condition holds for all $j \in I \backslash \{i\}$, i is fully aware and all agents have correct beliefs, then all agents a.s. survive.

Our first result (i) shows that whenever an agent is the only one in the economy capable of distinguishing and thus, trading between some relevant contingencies, he survives regardless of his beliefs, and regardless of the awareness of the other agents. This result is of special interest in view of Proposition 4 above. In particular, consider two sets of agents $\mathcal{I}$ and $\mathcal{J}$ who are exposed to some idiosyncratic risk, such as labor income in two different sectors of the economy. While each type of agent is aware of their own labor income stream, they are not aware of the variation in the income of the other group. Thus, mutual insurance of the idiosyncratic risk, which requires conditioning on the specific variation of income of each of the groups is impossible. Despite the lack of insurance against idiosyncratic risk, provided that the difference in payoffs in the relevant states is bounded away from 0 in the limit, the consumption of both types will be strictly positive i.o. on almost every path and hence, they will both survive a.s., regardless of their beliefs.

Finally, (ii) concerns the case in which a fully aware agent with correct beliefs is present in the economy. By Remark 10 (ii), this will cause all partially aware agents with incorrect beliefs to vanish a.s. However, as long as the awareness partitions are non-nested, and the partially aware agents have correct beliefs, they survive a.s..

5 Can Bounded Awareness be Welfare-Improving?

In this section we examine whether an agent might find it beneficial to remain unaware and thus implicitly constrain his trades to a partition of the state-space, and whether partial awareness and the resulting constraints on trade can be welfare-improving for society as a whole. The latter proposition is supported by Blume et al. (2018), who show that market incompleteness can improve the efficiency of the equilibrium allocation under the true probability distribution. Posner and Weyl (2013) discuss the welfare cost of financial speculation when traders hold heterogeneous beliefs and argue that restrictions on trades (such as, for example, a tax on transactions) might be necessary to alleviate these problems. (See also Weyl, 2007)

We first remark that, everything else being equal, less aware agents enjoy lower overall utility.

Proposition 13 *Consider an economy with differential awareness and assume that agent i is less aware than j, and thus, Ω^i is coarser than Ω^j. Suppose that the two agents have identical endowments, utility functions, and identical beliefs π restricted to Ω^i. In any equilibrium of the economy with equilibrium allocation c, $V_0^j\left(c^j\right) \geq V_0^i\left(c^i\right)$.*

Ceteris paribus, an agent who is less aware will be able to invest conditional on a coarser partition and will, as a result, obtain a lower welfare in equilibrium. Intuitively, the less aware agent has access to a larger set of trades that he can engage in and will obtain a higher utility from consumption. Note, however, that the weak inequality cannot be replaced by a strict one. For example, if i and j are the only agents in the population, no trade will occur in equilibrium and their welfare will be identical.

Proposition 13 seems to suggest that increasing an agent's awareness is always beneficial. This conclusion can however be misleading. Suppose for a moment that both i and j's beliefs on Ω^i are correct. While this assumption uniquely determines i's beliefs, it does not restrict j's beliefs on those events, which are not measurable with respect to Ω^i. Indeed, j's higher awareness forces him to formulate beliefs on a larger algebra of events. If precise probabilistic information about the likelihood of these events is not easily available, j's beliefs on such events might be wrong.

In what follows, we start by formalizing in Proposition 14 the sense in which a finer awareness partition leads to an increase of the K-L distance between agent's beliefs and the truth. This, in turn, has an effect on the agents' survival. In particular, as we show below in Example 15, it is easy to construct economies in which i a.s. survives, but j a.s. vanishes, his consumption converging to 0. Finally, if j's beliefs are not correct, $V_0^j\left(c^j\right)$ is a biased estimate of j's expected utility. Example 16 shows that in this case j's expected utility with respect to the truth may be strictly lower than that of i.

We start by examining how a change in an agent's awareness affects the distance of his beliefs relative to the truth.

Proposition 14 *Consider agents i and j such that i's awareness is coarser than j's, with probability beliefs that coincide on the coarser partition W^i. The K-L distance between j's beliefs on the more refined partition W^j and the truth is at least as great as the K-L distance of the beliefs of the two agents on the less refined partition W^i and the truth.*

Proposition 14 shows that the survival index with refined awareness is less than the survival index with coarse awareness. In the previous section, we showed that the K-L distance between the agent's beliefs and the truth is relevant for survival. In particular, an agent whose awareness increases will have to form beliefs over a new set of contingencies he did not consider before. Unless reliable statistical information is easily available and can be directly incorporated into the agent's decision, this opens room for mistakes. Even if the agent's beliefs on the coarser partition were correct, increased awareness may lead to wrong beliefs and thus potentially diminish his chances for survival.

Propositions 13 and 14 highlight a potential conflict: increasing awareness allows the agent to expand his trading opportunities and obtain higher welfare, but exposes him to losses due to wrong beliefs and eventually to the risk of vanishing, his consumption being reduced to 0 in the long-run.

To illustrate this result, consider the following example:

Example 15 *Consider an economy with a state space S. Let the set of agents be $I = \{1, 2, 3\}$. Agents 1 and 2 are fully aware so that $\Omega^1 = \Omega^2 = \Omega$. Agent 3 has a coarser awareness partition, W^3, which induces the partition Ω^3 of Ω. Suppose that all agents have correct i.i.d. beliefs on the coarsest of the three partitions, W^3, and thus on Ω^3. Assume as well that agent 1 has correct i.i.d. beliefs on S, whereas agent's 2 beliefs on S are wrong. Thus, the K-L distance of 2's beliefs from the truth is larger than that of 1, while their awareness partitions are identical. By Remark 10, part (i), agent 2 vanishes almost surely.*

Consider next the issue of survival for agents 1 and 3. Given the assumptions made above, Lemma 3 in Guerdjikova and Quiggin (2019 a, p. 1731) implies that π-a.s. on Ω^3,

$$\lim_{t\to\infty} \frac{\sum_{\sigma\in\omega^3} \pi\left(\sigma \mid \omega_t^3\right) u_1'\left(c^1(\sigma)\right)}{u_3'\left(c^3\left(\omega_t^3\right)\right)} = \frac{u_1'\left(c^1(\sigma_0)\right)}{u_3'\left(c^3(\sigma_0)\right)} \in (0, \infty) \tag{4}$$

We will show that agent 3 almost surely survives. Indeed, assume in a manner of contradiction that 3 vanishes on some ω, $\lim_{t\to\infty} c^3\left(\omega_t^3\right) = 0$ and thus, by Assumption 1,

$$\lim_{t\to\infty} u_3'\left(c^3\left(\omega_t^3\right)\right) = \infty.$$

To ensure that the ratio of marginal utilities in (4) is not 0, it is necessary that

$$\lim_{t\to\infty} \sum_{\sigma\in\omega^3} \pi\left(\sigma \mid \omega_t^3\right) u_1'\left(c^1(\sigma)\right) = \infty$$

and thus, by Assumption 1, that there exist an event $F \subseteq \omega$ with $\pi(F \mid \omega) > 0$ and $c^1(\sigma) = 0$ for all $\sigma \in F$. It follows that on F, π-a.s., $\lim_{t\to\infty}\left[c^1(\sigma_t) + c^3(\sigma_t)\right] = 0$ and since, by Assumption 2, the initial endowment of the economy is uniformly bounded away from 0, it follows that agent 2 cannot vanish π-a.s. on F, since on F, his consumption has to satisfy π-a.s. $\lim_{t\to\infty} c^2(\sigma_t) \geq m > 0$. But since agent 2 vanishes a.s., it follows that the probability of an ω on which agent 3 vanishes has to be 0.

We conclude that agent 3 survives π-a.s. By equation (4) this implies that π-a.s. on Ω^3, $\lim_{t\to\infty} \sum_{\sigma\in\omega^3} \pi\left(\sigma \mid \omega_t^3\right) u_1'\left(c^1(\sigma)\right) \neq \infty$ and thus, $c^1(\sigma) > 0$ π-a.s. on ω. Thus, agent 1 π-a.s. survives.

Comparing agents 2 *and* 3 *provides an illustration of Proposition 14. Both agents' beliefs on W^3 and thus on Ω^3 are correct. However, agent 2 is more aware than 3 and thus has to also form beliefs on the finer state space S. These beliefs happen to be wrong and thus the K-L distance of 2's beliefs from the truth is larger than that of 3, as stipulated in Proposition 14. The presence of an agent who is equally aware as 2, but who has correct beliefs, gives 2 the possibility to trade on events not-measurable with respect to Ω^3. While, as shown in Proposition 13, these additional trading opportunities in general increase the agent's welfare, trading on incorrect beliefs eventually leads to agent 2 vanishing. In contrast, the less aware agent 3 survives and enjoys strictly positive consumption in the limit.*

Example 15 illustrates the trade-off between higher expected utility resulting from higher levels of awareness and survival using two agents who are identical in all other characteristics but their awareness partitions, showing that higher levels of awareness may impede survival.

We next tackle the question of how the actual welfare of a single agent is impacted when the agent's awareness increases. To facilitate understanding, we consider a one-period economy with logarithmic preferences:

Example 16 *Consider a one-period economy. Assume that agent i's endowment is constant across states, $e^i(s) = e^i(s')$ for all s, $s' \in S$ and let all agents have logarithmic preferences.*

If i is fully aware, i's demand is given by:

$$c^i(s) = \frac{e^i \pi^i(s)}{p(s)}$$

where π^i are his beliefs, which are not necessarily correct, and we use the normalization $\sum_{s\in S} p(s) = 1$.

Next consider the case of i being only aware of the trivial partition $W^i = \{\{S\}\}$. In the one-period economy, this precludes any trade on the side of i and thus, i's consumption coincides with his initial endowment e^i.

Comparing i's equilibrium expected utility with respect to the truth in these two scenarios, we obtain that i will obtain a higher expected utility when more aware if and only if:

$$\sum_{s\in S} \pi(s) \ln c^i(s) > \ln e^i$$

which can be rewritten as:

$$-\sum_{s\in S} \pi(s) \ln \frac{\pi(s)}{\pi^i(s)} + \sum \pi(s) \ln \frac{\pi(s)}{p(s)} > 0. \qquad (5)$$

To understand the condition, note that the first term in (5) is the negative of the K-L distance of i's beliefs π^i with respect to the truth and is always non-positive. It does not depend on the initial endowment of the economy, nor on the beliefs of the other agents. The second term is the relative entropy of the pricing kernel $p(s)$ with respect to the truth and is always non-negative. Hence, whether i's expected utility will be higher when he is fully unaware depends on whether the deviation of his beliefs from the truth upon becoming fully aware exceeds the deviation of the equilibrium price kernel from the true probability. In particular, if other traders' beliefs are even further away from the truth than his own, i's expected utility will increase as he becomes fully aware.

However, if the other traders in the economy have correct beliefs and if the economy faces no aggregate risk, then $p(s)$ will be a convex combination of i's beliefs and the truth and hence, the entire term will be strictly negative. Hence, in an economy with no aggregate risk, i would be better off remaining unaware when all other agents have correct beliefs.

Furthermore, if the aggregate risk and i's initial endowment are both relatively small, so will be the second term on the l.h.s.,

$$p(s) \approx \pi(s)$$

and hence, i's expected utility will be higher if i is unaware.
If i's beliefs when fully aware assign 0-probability to a possible state, for example if $\pi^i(s_1) \to 0$,

$$-\sum_{s \in S} \pi(s) \ln \frac{\pi(s)}{\pi^i(s)} \to -\infty,$$

whereas $p(s_1) \neq 0$ will hold as long as at least one other trader deems s_1 possible. It follows that as a function of i's beliefs the second term is bounded above, whereas the first term is unbounded. We can conclude that for a given initial endowment of the economy, we can find sufficiently 'wrong' beliefs for the fully aware i such that he obtains higher expected utility when fully unaware. Alternatively, if for a given initial endowment, the chance of i's acquiring correct beliefs upon becoming aware is sufficiently low, he would be better off with a lower level of awareness.
Finally, note that in the case of logarithmic preferences in an infinite-horizon economy, the same argument can be applied to each time period. It allows us to identify conditions under which, in each period, i's expected utility with unawareness exceeds that with full awareness.

Example 16 shows that when the agent is not likely to acquire correct beliefs upon becoming aware of finer contingencies, he might be better off at a lower level of awareness. Intuitively, partial awareness restricts his investment opportunities and thus prevents him from trading on wrong beliefs and allows him to obtain a higher discounted expected utility relative to the true probability process. Such implicit restrictions on trade due to bounded awareness might also be beneficial from the point of view of the society as a whole.

Indeed, recent papers by GSS (2014) and Brunnermeier, Simsek and Xiong (2014) show that in markets with heterogeneous beliefs, the standard concept of Pareto-efficiency might not be appropriate. GSS (2014) argue that trades due to differences in beliefs have to be scrutinized more closely to determine whether they are truly Pareto-improving, or merely due to speculation. GSS (2014) propose a criterion which they refer to as a No-Betting Pareto (NBP) improvement, which requires that trades be a Pareto improvement for some possible belief. Brunnermeier et al. present the closely related idea of belief-neutral efficiency from the perspective of a social planner who knows that some agents have distorted beliefs (this is obvious from the fact of disagreement) but does not know the correct belief. An allocation is belief-neutral efficient (inefficient) if it is efficient (inefficient) under any convex combination of agents' beliefs.

Blume et al. (2018) point out that while equilibria in complete markets are Pareto-efficient, with heterogeneous beliefs, an equilibrium need not be Pareto-efficient with respect to the true probability distribution, that is, need not be true-Pareto-efficient. From this point of view, equilibria in incomplete markets may true-Pareto-dominate equilibria in complete markets with heterogeneous beliefs

In this section, we will focus on the NBP-improvement criterion and on true-Pareto-efficiency in order to understand the impact of differential awareness on welfare.

In what follows, we will use the insight that an increase in agents' awareness can lead to belief heterogeneity and result in some agents having wrong beliefs. We will generalize the welfare comparison obtained for a single agent to a welfare comparison of the equilibrium allocations across economies with different levels of awareness. For the purposes of the analysis, we will assume that for each agent i, Ω^i coincides with the finest partition of Ω, on which i is able to form correct beliefs. That is, if i were to become aware of a partition which is finer than Ω^i, his beliefs would be wrong. We will also assume, that there is at least one agent in the economy with correct beliefs on Ω. We will compare the equilibria in two economies: the economy in which all agents are fully aware and the economy in which each individual i is aware only of the finest partition Ω^i, on which his beliefs are correct and thus, final consumption is restricted to be measurable relative to Ω^i. We will refer to the former as the 'full awareness' economy, and to the latter as the 'bounded awareness' economy.

The following results show that the intuition obtained from example 16 applies more generally.

Proposition 17 *Suppose that the economy faces no aggregate risk, that is, $e(\sigma_t) = e(\sigma'_t)$ for all σ_t, $\sigma'_t \in \Omega_t$ and all t. Then the equilibrium of the "bounded awareness" economy is an NBP-undominated allocation, in the sense of GSS (2014) (where the relevant common probability distribution can be taken to be the truth), whereas the equilibrium of the "full awareness" economy is not a strict NBP-improvement over that of the "bounded awareness" one. The equilibrium of the "bounded awareness" economy is also true-Pareto efficient in* *the sense of Blume et al. (2018). If at least one agent has incorrect beliefs on Ω, when aware of Ω, the equilibrium of the "full awareness" economy is not true-Pareto-efficient and is not an NBP-improvement over that of the "bounded awareness" one.*

Proposition 18 *Suppose that $S = E \times G$ such that $e(\sigma_t; (e; g)) = e(\sigma'_t; (e; g'))$ for all g, $g' \in G$ and all σ_t, $\sigma'_t \in \Omega_t$. If every agent assigns correct probability to the partition of Ω generated by E, but at least one agent would have incorrect beliefs when aware of Ω, then the results of Proposition 17 apply.*

Proposition 17 in particular implies that, if the economy faces no aggregate risk, an agent, whose beliefs on Ω would be incorrect if he were aware of the full state space would be better off in the equilibrium of the "bounded awareness" economy than in the equilibrium of the "full awareness" one. Our next proposition examines the case in which the economy exhibits some aggregate risk.

Proposition 19 *Consider a regular economy, as defined by Kehoe and Levine (1985), with no aggregate risk and initial endowments $\left(e^j\right)_{j \in I}$ such that $e^i(\sigma_t) = e^i(\sigma'_t)$ for all σ_t, $\sigma'_t \in \Omega_t$ and all t. If agent i has incorrect beliefs, then there exists an open neighborhood B of $\left(e^j\right)_{\substack{j \in I \\ j \neq i}}$ such that the equilibrium of the "full awareness" economy with initial endowments $\left(\left(e^j\right)_{\substack{j \in I \\ j \neq i}} \in B, e^i\right)$ is not true-Pareto-efficient. Furthermore, for agent i, the expected utility of consumption in the equilibrium of the "bounded awareness" economy exceeds the expected utility of consumption with respect to the truth in the equilibrium of the "full awareness" economy.*

To understand Proposition 19 consider an agent i who starts with bounded awareness, correct beliefs on Ω^i and an initial endowment which completely insures him against risk, but would entertain potentially wrong beliefs if he were to become aware of Ω,. If the aggregate risk in the economy is sufficiently small, then i will be better off at the level of awareness Ω^i, which restricts his trades to the partition on which he has correct beliefs.

Finally, in economies with aggregate risk, if beliefs under full awareness are sufficiently far away from the truth, we can show that the equilibrium of the "full awareness" economy allocation is not NBP and that i's individual utility would be higher in the "bounded awareness" economy. This is in particular the case when i's beliefs on Ω (were he aware of it) are so far away from the truth that his equilibrium consumption is non-comonotonic with respect to the total initial endowment of the economy.

Proposition 20 *Suppose that the "full awareness" economy has an equilibrium, in which for two states σ_t and $\sigma'_t \in \omega^i_t$ for some i, we have $e(\sigma_t) \geq e(\sigma'_t)$ and $c^i(\sigma_t) < c^i(\sigma'_t)$. Then the equilibrium allocation in the "full awareness" economy is not NBP, nor is it true-Pareto efficient. Furthermore, suppose that* *all $j \neq i$ have correct beliefs, $\pi^i(s) = \pi(s)$ for all $s \notin w^i$ for some $w^i \subseteq S$, for the corresponding Ω^i, $e^i(\sigma) = e^i(\sigma')$ for all $\sigma \in \omega^i$ for some ω^i and the condition*

$$e(\sigma_t; s) \geq e(\sigma_t'; s') \text{ iff } c^i(\sigma_t; s) < c^i(\sigma_t'; s') \text{ for all } \sigma_t, \sigma_t' \text{ and all } s, s' \in w^i$$

holds. Then the equilibrium of the "full awareness" economy is not an NBP improvement over the equilibrium in which $W^i = \left\{\{s\}_{s \in S \setminus w^i}; w^i\right\}$ and i is strictly better off under the truth in the equilibrium of the "bounded awareness" economy than in the equilibrium of the "full awareness" economy.

In the light of these propositions, an agent who optimizes without regard to the possibility of constrained awareness and incorrect beliefs may be described as 'naively unaware'[8]. An important example of naive unawareness is that of agents who delegate their trading strategies to a computer program, designed to produce optimal outcomes for a formally specified model of the world. In the absence of external intervention, trading strategies of this kind cannot take account of possibilities that are excluded by design. This case is of particular interest since it is possible (at least in principle) for an external observer to determine the agent's model by inspection of the relevant computer code.

By contrast, a boundedly rational but sophisticated agent, who understands propositions such as 20 may prefer a bounded awareness equilibrium.

In the model presented here, agents' awareness is exogenously given. Hence, even if agents understand that they would be better off in a bounded awareness equilibrium, they cannot choose to bound their own awareness. They can, however, adopt robust heuristic procedures rather than attempting to optimize on the basis of beliefs that may be incorrect, see Gigerenzer (2007). We consider this issue in the next section.

6 Consciousness of Unawareness and Heuristics

The link between bounded awareness and heuristic constraints is developed by Grant and Quiggin (2013 b). In the model of Grant and Quiggin (2013 a) agents cannot be aware, in the modal-logical sense, of their own unawareness, but may nonetheless infer on the basis of induction from experience that their model of the world is incomplete and will be subject to unforeseen future surprises. Agents may therefore choose to adopt heuristic constraints on their decisions, such as those associated with the 'precautionary principle' (Grant and Quiggin 2013 b).

Two heuristics seem particularly relevant

(a) portfolio constraints

(b) liquidity preference

Guerdjikova and Quiggin (2019 a) showed that, under some circumstances, agents may be better off when they are subject to exogenous constraints on the

[8] This terminology is particularly relevant in the case where incorrect beliefs arise from restricted awareness of the set of possible states of nature.

set of assets in which they are allowed to trade. Sophisticated but boundedly aware agents may choose to adopt constraints on their portfolio holdings, with the aim of reducing their vulnerability to incorrect beliefs. A heuristic of the general form 'don't trade assets you don't understand' would be one way to implement such a constraint.

Next we consider liquidity preference. A naively unaware agent will not consider the possibility that they will attain a more refined awareness in the future. Hence, if asset markets allow for contracts indefinitely into the future, they will be willing to adopt a complete contingent consumption plan, measurable with respect to their initial awareness at time $t = 0$. This plan will constrain subsequent opportunities for retrading of which they may become aware.

By contrast, sophisticated, but boundedly aware agents understand that their awareness may be refined over time, but cannot explicitly formulate this understanding. In these circumstances, agents may attach a 'liquidity premium' to their investment plans, restricting long-term investments in order to maintain the capacity to take advantage of future retrading possibilities.

The liquidity premium must be distinguished from an option value. Given the assumption that markets are effectively complete for all agents, the agent can replicate any option that can be described in terms of the state space available to them. A naively unaware agent will therefore not perceive any option value from maintaining an ability to retrade.

6.1 Ecological Rationality

Following Gigerenzer (2007) it is useful to apply the concept of ecological rationality to the heuristics described above. A heuristic is ecologically rational in a given environment if it yields better results, on average, than optimization based on an incomplete and possibly inaccurate model of that environment. Importantly, an agent cannot know that a heuristic is ecologically rational, since this would require them to possess a complete and accurate model. Only a fully aware external observer can make a definitive assessment. However, based on induction from experience, agents may adopt heuristics that have previously worked well in similar environments.

To illustrate, consider the heuristic constraint 'don't invest in the share market - there are always people smarter than you'. The Blume and Easley result implies that this heuristic is ecologically rational for all but the best informed agent.

In this paper, we strengthen this claim to incorporate differential awareness. In particular, the heuristic 'don't trade assets you don't understand' emerges as ecologically rational for all but the most aware and best informed agent.

7 Concluding Comments and Future Directions

In this paper, we have shown that relaxing the unrealistic assumption of unbounded awareness leads to a richer model of financial markets. In many respects, this model is more realistic than the standard model. Agents can survive even if their beliefs are inaccurate, and asset prices display both an equity premium and a low real bond rate. This model represents a starting point for a more comprehensive analysis of financial markets with differential awareness. In future work, we plan to develop the model in two directions.

First, the present paper considers only the case of coarse awareness and takes probability beliefs as given. We propose to model differences in probability beliefs using the concept of restricted awareness. To the extent that agents are unaware of some possible states of nature, they will underweight the probability events of which those states are elements, and overweight the probability of complementary events. This suggests that beliefs can be derived endogenously from assumptions about the agents' awareness structure.

Second, the analysis in this paper has focused on the welfare characteristics of equilibrium. However, financial markets are not always in equilibrium. In particular, financial crises are inherently disequilibrium situations, often involving a transition from one equilibrium to another, Pareto-inferior equilibrium. As recent experience shows, financial crises can arise when certain agents become aware of contingencies that were previously unknown to them and the probabilities of which are difficult to estimate correctly. We conjecture that such changes in awareness that lead to a reduction in the proportion of agents who can survive in the long term are likely to result in crises in which these agents are forced to liquidate their portfolios. Conversely, markets where most agents are constrained to trade on events on which their beliefs are correct enhance survival and may thus be less prone to crises.

Taken together, these extensions suggest the possibility of a model of financial markets in which interactions between sophisticated but boundedly aware agents give rise to incomplete risk sharing and the survival of some, but not all, agents with inaccurate beliefs. These models would allow for the possibility of bubbles, busts and financial crises.

8 Appendix

Proof of Proposition 3:

Note first that since $e(\sigma_t') > e(\sigma_t'')$, measurability of initial endowments implies that there is a j such that $\omega_t', \omega_t'' \in \Omega_t^j$, that is, j can distinguish σ_t' and σ_t''.

(i) follows directly from the requirement that i's equilibrium consumption c^i has to be measurable with respect to Ω^i.

To show (ii), suppose to the contrary that the equilibrium provided full insurance against idiosyncratic risk. We then have: $c^j(\sigma_t) = c^j(\sigma_t')$ for all $j \in I$. The measurability constraint of i further implies that $c^i(\sigma_t') = c^i(\sigma_t'')$. Furthermore, by the equilibrium f.o.c., and since agents' beliefs are correct, we should have for any j such that $\omega_t', \omega_t'' \in \Omega_t^j$,

$$\frac{\pi^i\left(\omega_t^i\right) u_i'\left(c^i\left(\sigma_t''\right)\right)}{\pi^i\left(\omega_t\right) u_i'\left(c^i\left(\sigma_t\right)\right)}=\frac{\pi\left(\omega_t^i\right)}{\pi\left(\omega_t\right)}=\frac{\pi\left(\omega_t'\right)+\pi\left(\omega_t''\right)}{\pi\left(\omega_t\right)}=\frac{\pi^j\left(\omega_t'\right) u_j'\left(c^j\left(\sigma_t'\right)\right)+\pi^j\left(\omega_t''\right) u_j'\left(c^j\left(\sigma_t''\right)\right)}{\pi^j\left(\omega_t\right) u_j'\left(c^j\left(\sigma_t\right)\right)}.$$

Since however, $e\left(\sigma_t'\right) \neq e\left(\sigma_t''\right)$, there must be a $k \neq i$ such that $\omega_t', \omega_t'' \in \Omega_t^k$ and $c^k\left(\sigma_t'\right) \neq c^k\left(\sigma_t''\right)$ and since, by assumption $c^k\left(\sigma_t\right)=c^k\left(\sigma_t'\right)$, we obtain:

$$\frac{\pi^k\left(\omega_t'\right) u_k'\left(c^k\left(\sigma_t'\right)\right)+\pi^k\left(\omega_t''\right) u_k'\left(c^k\left(\sigma_t''\right)\right)}{\pi^k\left(\omega_t\right) u_k'\left(c^k\left(\sigma_t\right)\right)} \neq \frac{\pi^k\left(\omega_t'\right)+\pi^k\left(\omega_t''\right)}{\pi^k\left(\omega_t\right)}=\frac{\pi\left(\omega_t'\right)+\pi\left(\omega_t''\right)}{\pi\left(\omega_t\right)}$$

in contradiction to the equilibrium f.o.c. above. Thus, full insurance against idiosyncratic risk cannot obtain in equilibrium.

(iii) When all agents have correct beliefs, an unbiased price $\frac{p(\omega_t)}{p\left(\omega_t^i\right)}$ will satisfy

$$\frac{p\left(\omega_t\right)}{p\left(\omega_t^i\right)}=\frac{\pi\left(\omega_t\right)}{\pi\left(\omega_t^i\right)}.$$

However, at such price, all agents would choose to be fully insured against idiosyncratic risk, in contradiction to the result shown in (ii).

Proof of Proposition 4:

Suppose that in an equilibrium of the economy with differential awareness, the total consumption of the agents in $\mathcal{J}$ in σ_t' is given by $\sum_{j \in \mathcal{J}} c^j\left(\sigma_t'\right)$. By measurability of consumption of agents in $\mathcal{J}$, we have

$$\sum_{j \in \mathcal{J}} c^j\left(\sigma_t'\right)=\sum_{j \in \mathcal{J}} c^j\left(\sigma_t''\right).$$

By market clearing, we have

$$\sum_{i \in \mathcal{I}}\left[e^i\left(\sigma_t'\right)-c^i\left(\sigma_t'\right)\right]=\sum_{i \in \mathcal{I}}\left[e^i\left(\sigma_t''\right)-c^i\left(\sigma_t''\right)\right]$$

and by measurability of initial endowments and consumption of agents in $\mathcal{I}$,

$$\sum_{i \in \mathcal{I}}\left[e^i\left(\sigma_t\right)-c^i\left(\sigma_t\right)\right]=\sum_{i \in \mathcal{I}}\left[e^i\left(\sigma_t''\right)-c^i\left(\sigma_t''\right)\right]=\sum_{i \in \mathcal{I}}\left[e^i\left(\sigma_t'\right)-c^i\left(\sigma_t'\right)\right].$$

Therefore, market clearing implies

$$\sum_{j \in \mathcal{J}}\left[e^j\left(\sigma_t\right)-c^j\left(\sigma_t\right)\right]=\sum_{j \in \mathcal{J}}\left[e^j\left(\sigma_t'\right)-c^j\left(\sigma_t'\right)\right].$$

Proof of Proposition 5:

(i) Since $\sigma_t, \sigma_t' \in \omega_t^i$, we have $c^i\left(\sigma_t\right)=c^i\left(\sigma_t'\right)$ in any equilibrium of the economy with differential awareness. For such an allocation to be an equilibrium of the full awareness economy satisfying assumptions 1 and 3 with endowment process e and with homogeneous beliefs $\tilde{\pi}$, we need

$$\frac{\tilde{\pi}(\sigma_t)\, u_i'(c_i(\sigma_t))}{\tilde{\pi}(\sigma_t')\, u_i'(c_i(\sigma_t'))} = \frac{\tilde{\pi}(\sigma_t)}{\tilde{\pi}(\sigma_t')} = \frac{\tilde{\pi}(\sigma_t)\, u_j'(c_j(\sigma_t))}{\tilde{\pi}(\sigma_t')\, u_j'(c_j(\sigma_t'))} \tag{6}$$

for any $j \neq i$. This implies $c_j(\sigma_t) = c_j(\sigma_t')$ for all $j \neq i$. But since $e(\sigma_t) > e(\sigma_t')$, this cannot be an equilibrium of the full awareness economy, thus proving the claim.

(*ii*) When beliefs $\tilde{\pi}^i$ and $\tilde{\pi}^j$ are allowed to be heterogeneous, condition (6) becomes:

$$\frac{\tilde{\pi}^i(\sigma_t)\, u_i'(c_i(\sigma_t))}{\tilde{\pi}^i(\sigma_t')\, u_i'(c_i(\sigma_t'))} = \frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma_t')} = \frac{\tilde{\pi}^j(\sigma_t)\, u_j'(c_j(\sigma_t))}{\tilde{\pi}^j(\sigma_t')\, u_j'(c_j(\sigma_t'))}$$

Together with $e(\sigma_t) > e(\sigma_t')$, this implies that there is a j such that $c^j(\sigma_t) > c^j(\sigma_t')$ and thus, by Assumption 1, $u_j'(c_j(\sigma_t)) < u_j'(c_j(\sigma_t'))$, or

$$\frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma_t')} < \frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma_t')}$$

as claimed.

(*iii*) The existence of an economy with i.i.d. beliefs for all agents implies just as in part (*ii*) that there exist two agents j and k (not necessarily distinct, but distinct from i) such that $c_j(\sigma_t, s) > c_j(\sigma_t, s')$ and $c_k(\sigma_{t'}', s) < c_k(\sigma_{t'}', s')$. We also have:

$$\frac{\tilde{\pi}^i(s)\, u_i'(c_i(\sigma_t, s))}{\tilde{\pi}^i(s')\, u_i'(c_i(\sigma_t, s'))} = \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} = \frac{\tilde{\pi}^j(s)\, u_j'(c_j(\sigma_t, s))}{\tilde{\pi}^j(s')\, u_j'(c_j(\sigma_t, s'))}$$

$$\frac{\tilde{\pi}^i(s)\, u_i'(c_i(\sigma_t', s))}{\tilde{\pi}^i(s')\, u_i'(c_i(\sigma_t', s'))} = \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} = \frac{\tilde{\pi}^k(s)\, u_k'(c_k(\sigma_t', s))}{\tilde{\pi}^k(s')\, u_k'(c_k(\sigma_t', s'))}$$

If $j = k$, then we obtain

$$\frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} < \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')} \text{ and } \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} > \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')},$$

a contradiction.

Suppose next that $j \neq k$ and more specifically that

$$\begin{aligned} c_k(\sigma_t, s) &\leq c_k(\sigma_t, s') \\ c_j(\sigma_{t'}', s) &\geq c_j(\sigma_{t'}', s') \end{aligned}$$

with at least one of the inequalities being strict (otherwise, we could restate the argument above for either j or k). Then, we obtain:

$$\frac{\tilde{\pi}^k(s)}{\tilde{\pi}^k(s')} \leq \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')}$$

and

$$\frac{\tilde{\pi}^k(s)}{\tilde{\pi}^k(s')} \geq \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')},$$

with at least one of the inequalities being strict, a contradiction.

Proof of Proposition 13:

Follows directly from the proof of Proposition 4, Guerdjikova and Quiggin (2019 a, p. 1708).

Proof of Proposition 14:

For any event $w^i \in W^i$ in the coarser partition, and a sub-event $w^{ij} \in W^j$, $w^{ij} \subseteq w^i$ in the finer partition write the true conditional probability as

$$\pi\left(w^{ij}|w^i\right) = \frac{\pi\left(w^{ij}\right)}{\pi\left(w^i\right)}$$

and the conditional probability implied by j's beliefs as

$$\pi^j\left(w^{ij}|w^i\right) = \frac{\pi^j\left(w^{ij}\right)}{\pi^j\left(w^i\right)} = \frac{\pi^j\left(w^{ij}\right)}{\pi^i\left(w^i\right)}$$

Taking ln on both sides of the equation, summing over all $w^{ij} \subseteq w^i$, multiplying by $\pi\left(w^{ij}\right)$ and summing again over all $w^i \in W^i$, we obtain:

$$\begin{aligned}
\sum_{w^j \in W^{ij}} \pi\left(w^j\right) \ln \frac{\pi\left(w^j\right)}{\pi^j\left(w^j\right)} &= \sum_{w^i \in W^i} \sum_{w^{ij} \subseteq w^i} \pi\left(w^{ij}\right) \ln \frac{\pi\left(w^{ij}\right)}{\pi^j\left(w^{ij}\right)} \\
&= \sum_{w^i \in W^i} \pi\left(w^i\right) \sum_{w^{ij} \subseteq w^i} \pi\left(w^{ij}|w^i\right) \ln \frac{\pi\left(w^{ij}\right)}{\pi^j\left(w^{ij}\right)} \\
&= \sum_{w^i \in W^i} \pi\left(w^i\right) \sum_{w^{ij} \subseteq w^i} \pi\left(w^{ij}|w^i\right) \left(\ln \frac{\pi\left(w^{ij}|w^i\right)}{\pi^j\left(w^{ij}|w^i\right)} + \ln \frac{\pi\left(w^{ij}\right)}{\pi^j\left(w^{ij}\right)} \right) \\
&= \sum_{w^i \in W^i} \pi\left(w^i\right) \ln \frac{\pi\left(\omega_i\right)}{\pi^i\left(\omega_i\right)} + \sum_{w^i \in W^i} \sum_{w^{ij} \subseteq w^i} \pi\left(w^{ij}\right) \left(\ln \frac{\pi\left(w^{ij}|w^i\right)}{\pi^j\left(w^{ij}|w^i\right)} \right) \\
&\geq \sum_{w^i \in W^i} \pi\left(w^i\right) \ln \frac{\pi\left(w^i\right)}{\pi^i\left(w^i\right)}
\end{aligned}$$

which completes the proof.

Proof of Proposition 17:

Note that when beliefs of each agent i are correct on his partition Ω^i and there is no aggregate risk, the equilibrium of the "bounded awareness" economy provides full insurance to all agents. This equilibrium is thus both Pareto-efficient and by Theorem 2 of GSS (2014) undominated by NBP. Hence, if all agents' beliefs on Ω are correct, the equilibrium of the "bounded awareness" economy coincides with that of the "full awareness" economy and is thus both true-Pareto-efficient and NBP-efficient. In contrast, when there is at least one agent with incorrect beliefs on Ω and thus, (since an agent with correct beliefs on Ω is assumed to exist), at least two agents with distinct beliefs, full insurance no longer obtains in the equilibrium of the "full awareness" economy. Hence, the equilibrium allocation of the "full awareness" economy is a bet relative to the equilibrium allocation of the "bounded awareness" economy and by Proposition 2 of GSS (2014), the former does not (even weakly) NBP-dominate the latter. Since the full insurance equilibrium allocation in the "bounded awareness" economy would also obtain if all agents were to have correct beliefs, it is also a true-Pareto-efficient allocation as in Blume et al. (2018). In contrast, as shown by Blume et al. (2018), the equilibrium in the "full awareness" economy is not true-Pareto-efficient, whenever there is at least one agent with incorrect beliefs.

Proof of Proposition 18:

The result follows by combining the proof of Proposition 17 with the discussion in Section 4.5 in GSS (2014).

Proof of Proposition 19:

We know from Proposition 17 that in the economy with no aggregate risk, agent i who has incorrect beliefs is strictly worse off with respect to the truth in the equilibrium of the "full awareness" economy than in that of the "bounded awareness" economy, in which he obtains a utility at least as large as that of his initial endowment. If the economy is regular in the sense of Kehoe and Levine (1985), there is an open set of economies close to the economy with no aggregate risk such that on this set, the equilibrium allocation depends continuously on the initial endowments. Hence, we can find a sufficiently small set of initial endowments such that i is strictly worse off with respect to the truth in the equilibrium of the "full awareness" economy than if he consumed his initial endowment. Since he would be at least as well off with respect to the truth as consuming his initial endowment in the "bounded awareness" economy, the statement of the proposition obtains.

Proof of Proposition 20:

Suppose that for two states σ_t and $\sigma'_t \in \omega^i_t$ for some i, we have $e\left(\sigma_t\right) \geq e\left(\sigma'_t\right)$ and $c^i\left(\sigma_t\right) < c^i\left(\sigma'_t\right)$. Then, there is an agent $j \in I$ such that $c^j\left(\sigma_t\right) > c^j\left(\sigma'_t\right)$. For some probability distribution on Ω, $\hat{\pi}$, define consumption plan $\tilde{c}^i$ for i and $\tilde{c}^j$ for j as:

$$\begin{aligned}\tilde{c}^i\left(\sigma_t\right) &= \tilde{c}^i\left(\sigma'_t\right) = E_{\hat{\pi}}\left(c^i \mid \{\sigma_t; \sigma_{t'}\}\right)\\ \tilde{c}^j\left(\sigma_t\right) &= \tilde{c}^j\left(\sigma'_t\right) = E_{\hat{\pi}}\left(c^j \mid \{\sigma_t; \sigma_{t'}\}\right)\end{aligned}$$

and $\tilde{c}^i\left(\sigma''_t\right) = c^i\left(\sigma''_t\right)$, $\tilde{c}^j\left(\sigma''_t\right) = c^j\left(\sigma''_t\right)$ for all $\sigma''_t \notin \{\sigma_t; \sigma'_t\}$ and let

$$\begin{aligned}\epsilon^i &= c^i - \tilde{c}^i\\ \epsilon^j &= c^j - \tilde{c}^j.\end{aligned}$$

Since ϵ^i and ϵ^j are non-zero in only two states and have 0-expectation according to $\hat{\pi}$, they must be collinear, that is:

$$\epsilon^j = -\lambda\epsilon^i$$

for some $\lambda > 0$.

Suppose first that $\lambda \geq 1$. By construction, ϵ^i is independent of $\tilde{c}^i$. Similarly, ϵ^j, and hence, $-\epsilon^i$ is mean-independent of $\tilde{c}^j$. Taking ϵ^i away from i leaves him with a consumption plan $\tilde{c}^i$, which has the same expected consumption, but is less risky than c^i and hence, agent i is better off after the transfer. Giving ϵ^i to agent j leaves him with the plan $c^j + \epsilon^i = \tilde{c}^j + (\lambda - 1)\left(-\epsilon^i\right)$. Since $\lambda \geq 1$ and $\tilde{c}^j + \lambda\left(-\epsilon^i\right) = c^j$, c^j is more risky than $c^j + \epsilon^i$ and has the same expectation as $c^j + \epsilon^i$, hence agent j is also better off after the transfer.

If $\lambda < 1$, the transfer of ϵ^j to agent i makes both i and j better off under the truth.

Since the argument holds for any probability distribution $\hat{\pi}$ that is common to the agents, including the true one, it follows that the allocation c is not NBP, nor is it true-Pareto efficient.

To show the second part of the Proposition, note that if all $j \neq i$ have correct beliefs, then $c^j(\sigma_t) > c^j(\sigma_t')$ is satisfied in the unconstrained equilibrium for all $j \in I \backslash \{i\}$. Define ϵ^i and $\left(\epsilon^j\right)_{j \neq i}$ as above and let λ be defined as:

$$\lambda =: \frac{\sum_{j \neq i}\left[c^j(\sigma_t) - c^j(\sigma_t')\right]}{c^i(\sigma_t') - c^i(\sigma_t)} = \frac{e(\sigma_t) - e(\sigma_t') + c^i(\sigma_t') - c^i(\sigma_t)}{c^i(\sigma_t') - c^i(\sigma_t)};$$

Since

$$\begin{aligned}
\epsilon^i &=: \begin{pmatrix} c^i(\sigma_t) - \hat{\pi}(\sigma_t \mid \cdot)\, c^i(\sigma_t) - (1 - \hat{\pi}(\sigma_t \mid \cdot))\, c^i(\sigma_t'); \\ c^i(\sigma_t') - \hat{\pi}(\sigma_t \mid \cdot)\, c^i(\sigma_t) - (1 - \hat{\pi}(\sigma_t \mid \cdot))\, c^i(\sigma_t') \end{pmatrix} \\
&= \left((1 - \hat{\pi}(\sigma_t \mid \cdot))\left[c^i(\sigma_t) - c^i(\sigma_t')\right]; \hat{\pi}(\sigma_t \mid \cdot)\left[c^i(\sigma_t') - c^i(\sigma_t)\right]\right) \\
&= \left[c^i(\sigma_t') - c^i(\sigma_t)\right]\left(-(1 - \hat{\pi}(\sigma_t \mid \cdot)); \hat{\pi}(\sigma_t \mid \cdot)\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{j \neq i} \epsilon^j &=: \begin{pmatrix} e(\sigma_t) - c^i(\sigma_t) - \hat{\pi}(\sigma_t \mid \cdot)\left[e(\sigma_t) - c^i(\sigma_t)\right] - (1 - \hat{\pi}(\sigma_t \mid \cdot))\left[e(\sigma_t') - c^i(\sigma_t')\right]; \\ e(\sigma_t') - c^i(\sigma_t') - \hat{\pi}(\sigma_t \mid \cdot)\left[e(\sigma_t) - c^i(\sigma_t)\right] - (1 - \hat{\pi}(\sigma_t \mid \cdot))\left[e(\sigma_t') - c^i(\sigma_t')\right] \end{pmatrix} \\
&= \begin{pmatrix} (1 - \hat{\pi}(\sigma_t \mid \cdot))\left[e(\sigma_t) - e(\sigma_t') + c^i(\sigma_t') - c^i(\sigma_t)\right]; \\ \hat{\pi}(\sigma_t \mid \cdot)\left[e(\sigma_t') - e(\sigma_t) + c^i(\sigma_t) - c^i(\sigma_t')\right] \end{pmatrix} \\
&= \left[e(\sigma_t) - e(\sigma_t') + c^i(\sigma_t') - c^i(\sigma_t)\right]\left((1 - \hat{\pi}(\sigma_t \mid \cdot)); -\hat{\pi}(\sigma_t \mid \cdot)\right)
\end{aligned}$$

we have that

$$\sum_{j \neq i} \epsilon^j = -\lambda \epsilon^i$$

and, clearly, $\lambda \geq 1$. Hence, giving each $j \neq i$ a transfer $-\epsilon^j$ leaves each j with a plan which has the same expected consumption as c^j, but lower risk. This would leave:

$$\epsilon^i - \lambda \epsilon^i = (1 - \lambda)\, \epsilon^i = \frac{\lambda - 1}{\lambda} \sum_{j \neq i} \epsilon^j < \sum_{j \neq i} \epsilon^j$$

to be distributed. We can now allocate to each $j \neq i$ (in addition to $-\epsilon^j$), $\frac{\lambda-1}{\lambda}\epsilon^j$. Hence, each $j \neq i$ receives the allocation:

$$c^j - \epsilon^j\left(1 - \frac{\lambda - 1}{\lambda}\right) = \tilde{c}^j + \frac{\lambda - 1}{\lambda}\epsilon^j,$$

which has the same expectation as c^j and, since $\frac{\lambda-1}{\lambda} < 1$, is less risky than $c^j = \tilde{c}^j + \epsilon^j$. Hence, every agent in the economy is made strictly better off by the transfer. Clearly, the same construction can be repeated for any two σ_t and σ'_t for which i's equilibrium consumption is not comonotonic with the initial endowment.

Furthermore, since i's allocation after the transfer is measurable with respect to Ω^i, and since i's beliefs are correct on Ω^i, an equilibrium allocation of the economy with constraints on i's portfolio will be a Pareto-improvement with respect to any common beliefs (and in particular, with respect to the truth) over the allocation after transfers. It thus follows that the equilibrium of the "full awareness" economy is neither a NBP-improvement, nor a truth-Pareto-improvement over the equilibrium of the "bounded awareness" economy.

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